Decentralized Control of Multi-Agent Systems using Local Density Feedback

ACC 2021

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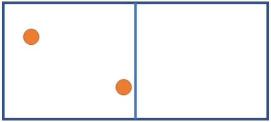
Joint work with K. Elamvazhuthi (UCLA) and S. Berman (ASU)

A simple game



- Distribute agents amongst two blocks
- Desired distribution $\mu^d = [0.5, 0.5]$.

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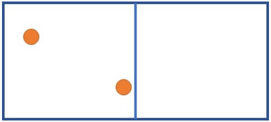
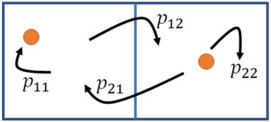


- Distribute agents amongst two blocks
- Desired distribution $\mu^d = [0.5, 0.5]$.
- Rules:
  - Each agent is <u>identity free</u>
  - <u>Decentralized</u> law, i.e. no central computer
  - No inter-agent communication

Solution

- Solution: Make it Stochastic!
- $P = \begin{bmatrix} p_{11} = H & p_{12} = T \\ p_{21} = H & p_{22} = T \end{bmatrix}$



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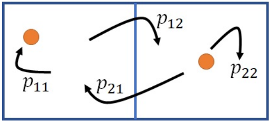

- Con: Agents keep transitioning!

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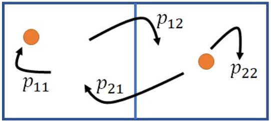

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Gist of Our Work

Control agent distribution when agents follow some Markov process

Mean-field model 101

- Graph based methods popular in multi-agent systems.

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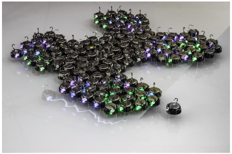


- Each agent follows an identical Markov process $\implies$ mean-field behavior determined by the *Kolmogorov forward equation*.
- Model is independent of agent population size, therefore, scalable.

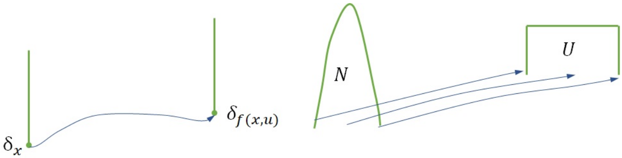
Alternative viewpoint

- Classical ODE (Deterministic) can be seen as evolution of Dirac delta function

$$\dot{x}(t) = f(x(t), u(t)).$$

- Markov process is evolution of distributions (pdfs):

$$\dot{\mu}(t) = F(\mu(t), k(t)).$$



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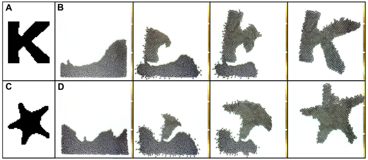
- States (μ) are probability distributions, transition rates/probabilities as controls k
- Stabilizability: System is stabilizable if given μ_d, there exists a map $k(\cdot)$ such that μ_d is asymptotically stable for (1)
- Challenge: $\mu \in \Sigma$ - infinite dimensional.

Potential applications

Redistributing large number of agents for environmental monitoring, surveillance, autonomous construction.



Harvard University/Michael Rubenstein

Goals

- Stabilizability - Existence of control law(s) k to stabilize desired distributions.
- Equilibrium of the Microscopic model - Zero state transitions at equilibrium to minimize energy expended by agents at equilibrium.

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Model

Discrete-Time, Continuous State-Space Markov Process ($\mathbb{R}^n$ or some manifold)
Ideal for modeling robots as robots don't evolve on graphs

Problem Formulation

Agent Model

$$x_{n+1} = F(x_n, u_n), \quad x_0 \in \Omega$$

- $x_n \in \Omega, \; u_n \in U, \; F : \Omega \times U \to \mathbb{R}^d$
- $(u_n)_{n=1}^{\infty} \in U$ such that $F(x_n, u_n) \in \Omega$

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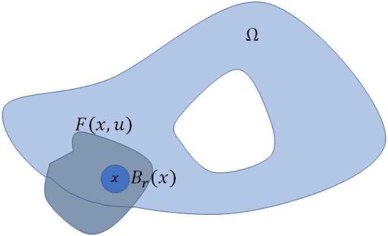

Assumptions

- Domain $\Omega \subset \mathbb{R}^d$ is closed, bounded, connected, $\delta\Omega$ is 'regular'
- Controls $U \subset \mathbb{R}^d$ is closed, bounded (compact)
- F is continuous, C^1, non-singular
- Locally controllable condition: there exists $r > 0$ such that, for every $x \in \Omega$, $B_r(x) \cap \Omega \subseteq F(x, U)$.

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- N - number of '*identity free* agents
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- $\frac{1}{N}\sum_{k=1}^{N}\delta_{x_n^k}$ not a state variable of the agent dynamics; in order to make it a state variable:
- Take $N \to \infty$

Problem Formulation I

Forward equation = Mean Field Model:

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Given $\mu_d \in \mathcal{P}(\Omega)$ and F can we construct an operator P such that

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Theorem

There exists a control law that stabilizes a class of μ_d!
Any μ_d with density $f_d, f_d^{-1} \in L^\infty(\Omega) = \{g : g \text{ is bounded a.e.}\}$.

Solution

- Transition kernel $\sim$ Transition Probability in discrete state space
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- For the discretized model, constructed optimal K to drive $||f_0 - f_d||_2 \to 0$ exponentially
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Problem Formulation II

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$$\lim_{n\to\infty} P^n \mu_0 = \mu_d,\ \ \mu_0 \in \mathcal{P}(\Omega)$$

$$P = I \text{ at } \mu_d \implies \text{ agents stop switching at } \mu_d$$

Forward equation = Mean Field Model:

$\mu_{n+1} = P(\mu_n)\mu_n, \quad \mu_0 \in \mathcal{P}(\Omega) =$ Probability measures on Ω
P is now nonlinear, function of the current distribution μ_n

Theorem

There exists a time-dependent control law that satisfies above conditions.
Any μ_d with density $f_d \in L^\infty(\Omega) = \{g : g(x) < \infty \text{ a.e.}\}$

Stochastic Feedback Law

Let $k : \Omega \times U \to [0,1]$ be in $L^\infty(\Omega \times U)$ and

$$k(x,u) \begin{cases} > 0 \text{ for } m\text{-a.e. } x \in \Omega, u \in U \text{ st. } F(x,u) \in \Omega; \\ = 0 \text{ otherwise;} \end{cases}$$

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Assuming $F(x,0) = x$ (can be generalized to $F(x, V(x)) = x$).

$$K_\mu(x,W) = a_{f_\mu}(x) \int_W k(x,u)du + (1 - a_{f_\mu}(x))\delta_0(W).$$

$$a_f(x) = \begin{cases} \frac{f(x) - f_d(x)}{f(x)} & \text{for } m\text{-a.e. } x \text{ if } f(x) - f_d(x) > 0; \\ 0 & \text{otherwise.} \end{cases}$$

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Closed-Loop control for the Mean-Field model.

Operator induced by K_μ

Define P via $K : \Omega \times \mathcal{B}(U) \to [0,1]$.

Definition (Forward operator (measures))

$P : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$

$$(P_\mu \mu)(E) = \int_\Omega \int_U \chi_E(F(x,u)) K_\mu(x, du) d\mu(x)$$

Note: $\chi_E(z) \begin{cases} = 1 & \text{if } z \in E \\ = 0 & \text{otherwise} \end{cases}$

Solution

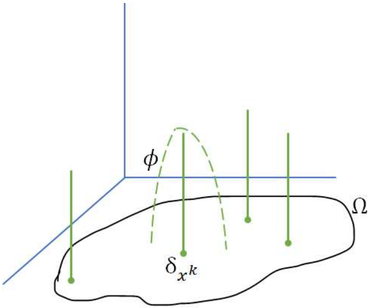
Theorem

If P preserves distributions with L^2 densities then f_d is globally asymptotically stable in the $L^1(\Omega, m)$ norm, i.e.

$$\|f_n - f_d\|_1 \to 0 \quad as \quad n \to \infty.$$

N-Agent System

- N agents on Ω evolving according to K.
- Empirical measure $m^N(x) = \frac{1}{N}\sum_{k=1}^{N}\delta_{x^k}$. $\implies$ Does not have a density
- Smoothen the measure by convolving with a Mollifer.



Mollifier

Standard bump function:

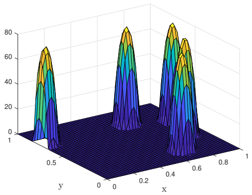
$$\phi(x) = \begin{cases} e^{-\left(\frac{1}{1-\|x\|^2}\right)}, & x \in (-1,1), \\ 0, & \text{otherwise.} \end{cases}$$

Change the support, for some $h > 0$

$$\phi_h(x) = h^{-2}\phi\left(\frac{x}{h}\right).$$

$\int \phi_h = 1$ for any h.

(Gif)

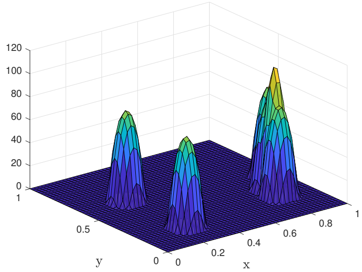


Mollifier

Convolution

Convolve the Dirac measure with a smooth (C^∞) function ϕ:

$$\phi * m^N = \int_\Omega \phi(x) dm^N = \frac{1}{N}\sum_{i=1}^{N} \phi(x - x^k).$$



Simulations: Mean Field Model & $N = 100$

Simulations: Mean Field Model & $N = 500$

Simulations: Mean Field Model & $N = 1500$

Agent Trajectories

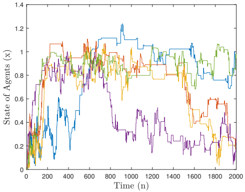


Figure: $N = 100$

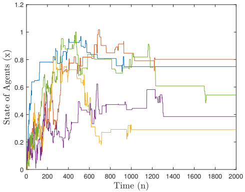


Figure: $N = 500$

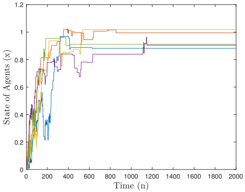


Figure: $N = 1000$

Thank you

Questions?

Shiba Biswal, Karthik Elamvazhuthi, and Spring Berman. "Target Distribution Stabilization Using Local Density Feedback for Multi-Agent Systems.", 2022. Accepted to *IEEE Transactions on Automatic Control.*

Shiba Biswal, Karthik Elamvazhuthi, and Spring Berman. "Stabilization of nonlinear discrete-time systems to target measures using stochastic feedback laws", 2021. *IEEE Transactions on Automatic Control.*

Shiba Biswal, Karthik Elamvazhuthi, and Spring Berman. "Stabilization of Multi-Agent Systems to Target Distributions using Local Interactions." Submitted to the 2020 *International Symposium on Mathematical Theory of Networks and Systems (MTNS)*, Cambridge, UK.

Shiba Biswal, Karthik Elamvazhuthi, and Spring Berman. "Fastest Mixing Markov Chain on a Compact Manifold." *IEEE Conference on Decision and Control (CDC)*, Nice, France, 2019.